SNAP Centre Workshop

Solving Systems of Equations

Introduction

When presented with an equation containing one variable, finding a solution is usually done using basic algebraic manipulation.

Example 1 $2(p-4)=8$ $\quad Solve\ for\ p.$

Dividing both sides of the equation by 2, then adding 4 gives us our solution.

$$\frac{2(p-4)}{2}=\frac{8}{2}$$

$$p-4\ +4=4+4$$

$$p=8$$

There is only one value p can have that makes our original equation true: 8

When presented with an equation with more than one variable, however, algebraic manipulation only gets us so far. It is possible to express one of the variables in terms of the other - effectively giving a set of solutions – but we can't find a single, definitive value for each variable.

Example 2 $4t+3r=1-r$ $\quad Solve\ for\ t.$

First, we want to isolate the term containing t. We do this by subtracting 3r from both sides of the equation.

$$4t+3r-3r=1-r-3r$$

$$4t=1-4r$$

Next, divide both sides by 4.

$$\frac{4t}{4}=\frac{1-4r}{4}$$

$$t=\frac{1}{4}-r$$

Our final result shows that there is not a single, unique value that can be found when solving for t, but rather that t depends on r; any value we substitute into the equation for r will result in a corresponding t value. Also, note that this set of solutions can be represented graphically by a line with a slope of -1, and a "t-intercept" of $(0,\frac{1}{4})$.

If we are presented with multiple equations containing some combination of the same variables (a **system of equations**), it is possible to find a single solution - if such a solution exists.

Substitution Method

The **substitution method** involves isolating a variable in one of the equations in a given system and substituting its value into the other given equations. The substitution process is repeated until it is possible to find a numerical value for one of the variables, at which point this value is used to solve for the values of the remaining variables, which ultimately results in a solution to the system of equations.

Example 3 $2a + 4b = 9 - a$ $\quad$ $Solve\ for\ a\ \&\ b.$

$$a = 3$$

Since a is given explicitly, we can substitute its value of 3 into our other equation immediately and solve for b.

$$2(3) + 4b = 9 - (3)$$

$$6 + 4b = 6$$

$$4b = 0$$

$$b = 0$$

Our solution to the given system as an (a, b) ordered pair is $(\mathbf{3}, \mathbf{0})$.

In **Example 3**, the numerical value of a is given explicitly. This will not always be the case.

Example 4 $2m - 3n = 7$ $\quad$ $Solve\ for\ m\ \&\ n.$

$$n = 3m + 1$$

In this example, we are given n in terms of m in the second equation. We can start by substituting this into our first equation in place of n.

$$2m - 3(3m + 1) = 7$$

Now we have one equation and one unknown.

$$2m - 9m - 3 = 7$$

$$-7m = 10$$

$$m = -\frac{10}{7}$$

We can now substitute the numerical value we found for m in either of our equations to solve for n. For this example, we will substitute into the second equation.

$$n = 3\left(-\frac{7}{10}\right) + 1$$

$$n = -\frac{21}{10} + 1$$

$$n = -\frac{21}{10} + \frac{10}{10}$$

$$n = -\frac{11}{10}$$

Our solution to the given system as an (m, n) *ordered pair is* $\left(-\frac{\mathbf{10}}{\mathbf{7}}, -\frac{\mathbf{11}}{\mathbf{10}}\right)$.

In **Example 4,** n was stated in terms of m. If this is not the case, it is necessary to pick a lone variable to isolate – as well as which equation to manipulate – before performing substitution.

Example 5 $5x + 2y - 1 = y - 3x$ $Solve\ for\ x\ \&\ y.$

$2x - 13 = 4x - y$

In the second equation in our system, we can see that y *is the lone variable. Given this, we will start by manipulating the second equation to express* y *in terms of* x.

$$2x - 13 - 4x = -y$$

$$-2x - 13 = -y$$

$$y = 2x + 13$$

We now substitute our rearranged second equation back into our first equation and solve for x.

$$5x + 2(2x + 13) - 1 = (2x + 13) - 3x$$

$$5x + 4x + 26 - 1 = -x + 13$$

$$9x + 25 = -x + 13$$

$$10x = -12$$

$$x = -\frac{12}{10} = -\frac{6}{5}$$

After simplifying, we find that $x = -\frac{6}{5}$*. We could plug this into either our first or second equation, but it would be even easier to substitute this value into our rearranged equation* $y = 2x + 13$*, so that's what we'll do.*

$$y = 2\left(-\frac{6}{5}\right) + 13$$

$$y = -\frac{12}{5} + 13$$

$$y = -\frac{12}{5} + \frac{65}{5}$$

$$y = \frac{53}{5}$$

Our solution to the given system as an (x, y) *ordered pair is* $\left(-\frac{\mathbf{6}}{\mathbf{5}}, \frac{\mathbf{53}}{\mathbf{5}}\right)$.

Elimination Method

The **elimination method** is another method for solving systems of equations that can prove to be less time consuming than the substitution method, particularly in systems that do not contain lone variables.

The process involves adding or subtracting *entire equations* to/from one another in order to eliminate all of the terms for one variable, while solving for the value of the other.

Example 7 $2x + 4y = 12$ $\quad Solve\ for\ x\ \&\ y.$

$5x + 4y = 16$

Noting that the y coefficients in both equations match, we can subtract our second equation from our first equation to eliminate our y terms.

$$(2-5)x + (4-4)y = 12 - 16$$

$$-3x = -4$$

$$x = \frac{4}{3}$$

Now that we have a value for x, we substitute it into either of the given equations to solve for y.

$$2\left(\frac{4}{3}\right) + 4y = 12$$

$$\frac{8}{3} + 4y = 12$$

$$4y = 12 - \frac{8}{3}$$

$$4y = \frac{36-8}{3}$$

$$4y = \frac{28}{3}$$

$$y = \frac{28}{12} = \frac{7}{3}$$

Our solution to the given system as an (x, y) ordered pair is $\left(\mathbf{\frac{4}{3}}, \mathbf{\frac{7}{3}}\right)$.

If the signs of the coefficients in front of the variable we wish to eliminate are opposite in the equations, we add the equations together instead of subtracting one from the other.

Example 8 $3s - 4t = 11$ $\quad Solve\ for\ s\ \&\ t.$

$-3s - 2t = 1$

Since our s coefficients vary only in sign, we can add the equations to cancel the terms.

$$(3-3)s + (-4-2)t = 11 + 1$$

$$-6t = 12$$

$t = -\frac{12}{6} = -2$

Again, we plug our t value back into either of our original equations to solve for s.

$3s - 4(-2) = 11$

$3s + 8 = 11$

$3s = 3$

$s = 1$

Our solution to the given system as an (s, t) ordered pair is $(\mathbf{1}, -\mathbf{2})$.

In **Examples 7 and 8**, the coefficients in front of one of the variables matched. This will not always be the case, but it is possible to manipulate the equations so that we may use the elimination method.

Example 9 $6m + 9n = 12$ $\quad$ $Solve\ for\ m\ \&\ n.$

$-5m + 3n = 7$

None of the coefficients in the given system of equations match. We could manipulate both equations so that the m coefficients match, however, it is possible to match n coefficients by only manipulating the second equation. If we multiply both sides of the second equation by 3, we will be able to cancel our n terms by subtracting our new equation from our first equation.

$3(-5m + 3n) = 3(7)$

$-15m + 9n = 21$

$(6 - (-15))m + (9 - 9)n = 21$

$9m = 21$

$m = \frac{21}{9}$

We will now substitute the value we have found for m back into our first equation to solve for n.

$6\left(\frac{21}{9}\right) + 9n = 12$

$\frac{126}{9} + 9n = 12$

$9n = 12 - \frac{126}{9} = 12 - \frac{42}{3}$

$9n = \frac{36}{3} - \frac{42}{3}$

$9n = -\frac{8}{3}$

$n = -\frac{8}{27}$

Our solution to the given system as an (m, n) ordered pair is $\left(\mathbf{\frac{21}{9}}, -\mathbf{\frac{8}{27}}\right)$.

No Solution and Infinite Solutions to a System of Equations

It is possible with the substitution method and the elimination method to find that there is no solution to a given system of equations, or there are infinitely many solutions to a given system.

Example 10 $p - 4q = 6$ $\quad$ $Solve\ for\ p\ and\ q.$

$$p - 12 = 6 + 12q - 2p$$

Since we can easily isolate p *in our first equation, we will use the substitution method.*

$$p = 6 + 4q$$

$$(6 + 4q) - 12 = 6 + 12q - 2(6 + 4q)$$

$$4q - 6 = 6 + 12q - 12 - 8q$$

$$4q - 6 = -6 + 4q$$

$$-6 = -6$$

When trying to solve for q*, all of our variables cancel, and we are left with a true statement. Any value we choose for our variables will have no bearing on whether or not the statement is true, therefore there are* ***infinitely many solutions****.*

Example 11 $3m - 6n = -7$ $\quad$ $Solve\ for\ m\ and\ n.$

$$-m + 2n = 10$$

Noticing that multiplying our second equation by 3 *allows us to add together the equations and cancel our* m *terms, we will attempt to use the elimination method.*

$$3(-m + 2n) = 3(10)$$

$$-3m + 6n = 30$$

We add this equation to our first equation to solve for n*.*

$$(3 - 3)m + (-6 + 6)n = -7 + 30$$

$$0 = 23$$

After adding our equations, all variables cancel and we are left with an untrue statement, indicating that there are no values we can use in place of our variables to make a true statement, and that there is ***no solution*** *to the given system.*

Solving Systems of Three Equations

Solving systems of three equations with three unknown variables involves the same concepts as solving systems of two equations. In fact, the goal when solving a system of three equations is to reduce the system down to two equations with two unknown variables.

Example 12 $2p + 3q - r = -1$ $\quad Solve\ for\ p, q, and\ r.$

$$-6p - 6q + 9r = 15$$

$$2p + 5q - r = 3$$

Our first step is using elimination to produce two separate equations where the same variable has been removed. We can start this by multiplying our first equation by 3 *and adding it to our second equation to produce what we will label equation* (1)*.*

$$(3(2) - 6)p + (3(3) - 6)q + (3(-1) + 9)r = 3(-1) + 15$$

$$(6 - 6)p + (9 - 6)q + (-3 + 9)r = -3 + 15$$

$$3q + 6r = 12$$

$$q + 2r = 4 \qquad Equation\ (1)$$

Next, we multiply our third equation and add it to our second equation to produce a second equation without any p *terms. We will label this equation (*2*)*

$$(3(2) - 6)p + (3(5) - 6)q + (3(-1) + 9)r = 3(3) + 15$$

$$(6 - 6)p + (15 - 6)q + (-3 + 9)r = 9 + 15$$

$$9q + 6r = 24$$

$$3q + 2r = 8 \qquad Equation\ (2)$$

Next, we solve our new system of equations. We can subtract (2) *from* (1) *to cancel* r *terms.*

$$(1 - 3)q + (2 - 2)r = 4 - 8$$

$$-2q = -4$$

$$q = 2$$

We now have a value for q *that we can plug back into either equation* (1) *or* (2) *to solve for* r*. We will use equation* (1)*.*

$$(2) + 2r = 4$$

$$2r = 2$$

$$r = 1$$

By substituting the values we have for r *and* q *back into our original system of equations, we can find a value for* p*, completing our solution. We will use the first equation to solve for* p*.*

$$2p + 3(2) - (1) = -1$$

$$2p + 6 - 1 = -1$$

$$2p = -6$$

$$p = -3$$

Our solution to the given system as an (p, q, r) ordered set is $(\mathbf{-3}, \mathbf{2}, \mathbf{1})$. We can verify that it is valid by plugging these values into any of our original equations for a true statement.

Solving Non-Linear Systems of Equations

Until this point, the systems of equations we have been finding solutions to have been composed of linear equations. Solving simple systems of non-linear equations is another useful skill to develop, and can be done using substitution.

Example 13 $\quad y = 6 \qquad Solve\ for\ x\ and\ y.$

$$y + x = x^2$$

Having been given y explicitly in our first equation, we can substitute its value into our second equation to solve for x.

$$(6) + x = x^2$$

We want to know the values of x for which this statement will be true. The easiest way to do this is by moving all of our terms to one side, and solving for the roots of the resulting quadratic equation.

$$0 = x^2 - x - 6$$

$$0 = (x + 2)(x - 3)$$

Knowing that $y = 6$, and $x = -2{,}3$, we can express our solution as the (x, y) ordered pairs $(\mathbf{-2}, \mathbf{6}), (\mathbf{3}, \mathbf{6})$.

Example 14 $\quad r = -4t + 2 \qquad Solve\ for\ r\ and\ t.$

$$r - t = t^2 - 4$$

Having r isolated in the first equation allows us to substitute it into the second equation as our first step.

$$(-4t + 2) - t = t^2 - 4$$

Just like before, we solve for the roots of the resulting quadratic equation to find our t values.

$$-5t + 2 = t^2 - 4$$

$$0 = t^2 + 5t - 6$$

$$0 = (t - 1)(t + 6)$$

We can substitute our t values, $t = 1, -6$, back into our first equation to solve for r.

For $t = 1$: $\quad r = -4(1) + 2$

$$r = -2$$

For $t = -6$ $\qquad r = -4(-6) + 2$

$$r = 24 + 2$$

$$r = 26$$

So, expressed as (r, t) *ordered pairs, our solution is* $(\mathbf{-2}, \mathbf{1}), (\mathbf{26}, \mathbf{-6})$***.***